On fixed points of infinite-dimensional generating function

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Outline

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Let $\{\boldsymbol{X}_i = (X_i^{(1)}, X_i^{(2)}, \cdots); i \geq 1\}$ be a sequence of random variables with values in $\mathbb{N}^\infty$. Denote the generating function of $\boldsymbol{X}_i$ by $F^{(i)}(\boldsymbol{s}) = \mathbb{E}\boldsymbol{s}^{\boldsymbol{X}_i}$.

Let $\boldsymbol{F}(\boldsymbol{s}) = (F^{(i)}(\boldsymbol{s}))_{i\geq 1}$. We are interested in fixed points set of $\boldsymbol{F}$ which is

$$T(\boldsymbol{F}) = \{\boldsymbol{s} \in [0,1]^\infty : \boldsymbol{F}(\boldsymbol{s}) = \boldsymbol{s}\}.$$

In this talk, we consider $\boldsymbol{F}(\boldsymbol{s})$ as an offspring generating function of a Galton-Watson process with a countable set of types (GWP-∞).

Infinite-dimensional generating function

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In this talk, we consider $\boldsymbol{F}(\boldsymbol{s})$ as an offspring generating function of a Galton-Watson process with a countable set of types (GWP-∞).

- A 1-type GWP $\{Z_n; n \geq 0\}$ satisfies:

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i,$$

where $\{\xi_i; i > 0\}$ is a sequence of i.i.d. random variables.

Let $h(s) = \mathbb{E}s^{\xi_1}$. If $h'(1) \leq 1$, $T(h) = \{1\}$; if $h'(1) > 1$, $T(h) = \{q, 1\}$, where q is the extinction probability which is the unique solution of $h(s) = s$ in $(0, 1)$.

- A d-type GWP $\{\boldsymbol{Z}_n = (Z_n^{(1)}, Z_n^{(2)}, \cdots, Z_n^{(d)}); n \geq 0\}$ satisfies:

$$\boldsymbol{Z}_n = \sum_{k=1}^{d} \sum_{i=1}^{Z_{n-1}^{(k)}} \boldsymbol{\xi}_{k,i}.$$

Let $f_i(\boldsymbol{s}) = \mathbb{E}\boldsymbol{s}^{\boldsymbol{\xi}_{i,1}}$ and $\boldsymbol{f}(\boldsymbol{s}) = (f_i(\boldsymbol{s}))_{1 \leq i \leq d}$. Denote the mean matrix by $\boldsymbol{A} = ((a_{ij}))$ with $a_{ij} = \frac{\partial f_i}{\partial s_j}(\boldsymbol{1})$.

Denote the maximal eigenvalue of $\boldsymbol{A}$ by ρ. If $\rho \leq 1$, $T(\boldsymbol{f}) = \{\boldsymbol{1}\}$; if $\rho > 1$, $T(\boldsymbol{f}) = \{\boldsymbol{q}, \boldsymbol{1}\}$, where $\boldsymbol{q}$ is the extinction probability which is the unique solution of $\boldsymbol{f}(\boldsymbol{s}) = \boldsymbol{s}$ in $(0,1)^d$.

• An infinite-type GWP (Moyal '62, Harris '63) $\{\boldsymbol{Z}_n = (Z_n^{(1)}, Z_n^{(2)}, \cdots); n \geq 0\}$ satisfies:

$$\boldsymbol{Z}_n = \sum_{k=1}^{\infty} \sum_{i=1}^{Z_{n-1}^{(k)}} \overline{\boldsymbol{\xi}}_{k,i},$$

where for any $k > 0$, $\{\overline{\boldsymbol{\xi}}_{k,i}; i > 0\}$ is a sequence of i.i.d. random variables with values in $l_1(\mathbb{N})$, where $l_1(\mathbb{N}) = \{\boldsymbol{x} \in \mathbb{N}^{\infty} : \mathbf{1} \cdot \boldsymbol{x} < \infty\}$.

Let $\boldsymbol{F}(\boldsymbol{s}) = (F_i(\boldsymbol{s}))_{i \geq 1}$ with $F_i(\boldsymbol{s}) = \mathbb{E}\boldsymbol{s}^{\overline{\boldsymbol{\xi}}_{i,1}}$. What about the $T(\boldsymbol{F})$ and its relation with the extinction probability?

Outline

Connection with BRW on an infinite graph

Let $P_i(\boldsymbol{j}) = \mathbb{P}(\overline{\boldsymbol{\xi}}_{i,1} = \boldsymbol{j})$. Then

$$F_i(\boldsymbol{s}) = \sum_{\boldsymbol{j} \in l_1(\mathbb{N})} P_i(\boldsymbol{j}) \boldsymbol{s}^{\boldsymbol{j}}.$$

GWPs-∞ can naturally be interpreted as branching random walks (BRWs) on an infinite graph where the types of particles correspond to the vertices of graph.

GWPs-∞ are of many applications as stochastic models for biological populations (Kimmel '02).

Connection with BRW on an infinite graph

Let $\boldsymbol{M} = ((m_{ij}))$ with $m_{ij} = \frac{\partial F_i}{\partial s_j}(\mathbf{1})$.

- The associated mean progeny representation graph of $\boldsymbol{M}$, irreducibility and connectivity. Assume non-singularity in each irreducible class henceforth.
- Constructing GWP-∞ to deal with stochastic models on an infinite graph. See Bertacchi and Zucca (2009) for edge-breading BRW.

Introduction for GWP-∞

The set of types to be infinite gives rise to three main challenges:

(1) First, as the mean progeny matrix $\boldsymbol{M}$ has infinite dimension, one has to look for a replacement to the spectral radius as an extinction criterion;

(2) Second, the concept of extinction has to be defined carefully: when there are infinitely many types, it is possible for every type to eventually disappear while the whole population itself explodes;

(3) Third, one needs to determine how to compute the extinction probability vector $\boldsymbol{q}$ which now has infinitely many entries.

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Challenge (1)

The reciprocal τ^{-1} of convergence radius τ of $\sum_{k\geq 0} r^k(\boldsymbol{M}^k)_{ij}$ is often used to replace the spectral radius of $\boldsymbol{M}$ in finite-type case, which is also called convergence norm of $\boldsymbol{M}$. The solutions $\boldsymbol{v}$ and $\boldsymbol{u}$ of $\tau\boldsymbol{v}\boldsymbol{M} = \boldsymbol{v}, \tau\boldsymbol{M}\boldsymbol{u} = \boldsymbol{u}$ are often called invariant measure and vector.

In infinite-dimensional cases, $\boldsymbol{v}$ and $\boldsymbol{u}$ may not exists or $\boldsymbol{v}\cdot\boldsymbol{u} = \infty$.

Ergodic property in types for GWP-∞

Ergodic property for infinite matrix (Seneta '81)

If the irreducible matrix $\boldsymbol{M}$ with convergence radius τ, satisfying

- $\sum_{k\geq 0}\tau^k(\boldsymbol{M}^k)_{ij}$ diverges: τ-recurrent;
- $\sum_{k\geq 0}\tau^k(\boldsymbol{M}^k)_{ij}$ converges: τ-transient.

If a τ-recurrent matrix $\boldsymbol{M}$ satisfies for some positive integers (i, j) and then for all,

- $\lim_{k\to\infty}\tau^k(\boldsymbol{M}^k)_{ij} > 0$: τ-positive recurrent;
- $\lim_{k\to\infty}\tau^k(\boldsymbol{M}^k)_{ij} = 0$: τ-null recurrent.

Ergodic property

The ergodic property of GWP-∞ in the typeset refers to the ergodic property of $\boldsymbol{M}$ (Sagitov '13).

Seneta (1981):

- If and only if GWP-∞ is transient, $\boldsymbol{v}$ and $\boldsymbol{u}$ do not exist;
- If and only if GWP-∞ is null recurrent, $\boldsymbol{v}$ and $\boldsymbol{u}$ exist but $\boldsymbol{v} \cdot \boldsymbol{u} = \infty$;
- If and only if GWP-∞ is positive recurrent, $\boldsymbol{v}$ and $\boldsymbol{u}$ exist and $\boldsymbol{v} \cdot \boldsymbol{u} < \infty$, moreover $\lim_{n\to\infty} \tau^n \boldsymbol{M}^n = \boldsymbol{u}\boldsymbol{v}$.

Challenge (2)

For a BRW on an infinite graph with initial configuration given by a single particle at a fixed vertex i, there are two kinds of survival:

- weak (or global) survival–the total number of particles in the graph is positive for all time;
- strong (or local) survival–the number of particles in vertex i is not eventually 0.

Extinction probability for GWP-∞

Given a typeset $\mathcal{T} \subset \mathbb{N}^+ = \{1, 2, 3, \cdots\}$, we can define the local extinction probability $\boldsymbol{q}(\mathcal{T}) = (q^{(i)}(\mathcal{T}))_{i \geq 1}$ as

$$q^{(i)}(\mathcal{T}) = \mathbb{P}(\lim_{n \to \infty} \sum_{l \in \mathcal{T}} Z_n^{(l)} = 0 | \boldsymbol{Z}_0 = \boldsymbol{e}_i).$$

- Global extinction probability: $\boldsymbol{q} = \boldsymbol{q}(\mathbb{N}^+)$;
- Partial extinction probability: $\tilde{\boldsymbol{q}} = \mathbb{P}(\{\text{Extinction for all finite typesets}\})$. In irreducible case, $\boldsymbol{q}(A) = \tilde{\boldsymbol{q}}$ for any finite typeset A.

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Challenge (3)

- For any typeset $\mathcal{T} \subset \mathbb{N}^+$, $\boldsymbol{F}(\boldsymbol{q}(\mathcal{T})) = \boldsymbol{q}(\mathcal{T})$.
- If $\inf_i q^{(i)} > 0$, then
  (1) $\mathbb{P}_i(|\boldsymbol{Z}_n| \to 0) + \mathbb{P}_i(|\boldsymbol{Z}_n| \to \infty) = 1$; (Jagers '92)
  (2) $\lim_{n\to\infty} \boldsymbol{F}_n(\boldsymbol{s}) = \boldsymbol{q}$ for any $\boldsymbol{s}$ with $\inf_i s_i < 1$, where $F_n^{(i)}(\boldsymbol{s}) = \mathbb{E}_i \boldsymbol{s}^{\boldsymbol{Z}_n}$. (Spataru '89)
- Assume the process is irreducible. If and only if $\tau^{-1} \leq 1$, $\tilde{\boldsymbol{q}} \leq \mathbf{1}$. If $\tau^{-1} > 1$, then $\boldsymbol{q} < \mathbf{1}$. (Bertacchi and Zucca '09)

Connections between Q and $T(\boldsymbol{F})$

Recall $T(\boldsymbol{F})$ is the set of fixed points of $\boldsymbol{F}(\cdot)$. Define the extinction probability set by $Q = \{\boldsymbol{q}(\mathcal{T}) : \mathcal{T} \subset \mathbb{N}^+\}$.

It is well-known that in finite-type cases, $\boldsymbol{q} = \tilde{\boldsymbol{q}}$. Either $Q = T(\boldsymbol{F}) = \{\mathbf{1}\}$ (in subcritical and critical case) or $Q = T(\boldsymbol{F}) = \{\boldsymbol{q}, \mathbf{1}\}$ (in supercritical case).

What about the relations between Q and $T(\boldsymbol{F})$ in infinite-type case?

Known results

Bertacchi and Zucca (2020) summarized the main known relations between Q and $T(\boldsymbol{F})$ in infinite-type case.

- $Q \subset T(\boldsymbol{F})$, $\tilde{\boldsymbol{q}} = \max\{Q\}$, $\boldsymbol{q} = \min\{Q\}$.
- There are examples for: (1) Q is uncountable; (2) Q is finite and $|Q| > 2$ while $T(F)$ is uncountable. (Spataru '89, Bertacchi and Zucca '17)
- Assume the process is irreducible and quasi-transitive. If $\boldsymbol{q}(\{i\}) < \mathbf{1}$ for some i, then $Q = \{\boldsymbol{q}, \mathbf{1}\}$. If $\boldsymbol{q}(\{i\}) = \mathbf{1}$ for all i, Q can be uncountable. $T(\boldsymbol{F})$ is unknown in both cases.
- In Lower Hessenberg GWP-∞, there are four cases for Q and $T(\boldsymbol{F})$.

Lower Hessenberg case

Braunsteins and Hautphenne (2017) proved the correctness of the conjecture in Lower Hessenberg GWP-∞, which assume type i particles can only produce type $j \le i+1$ particles.

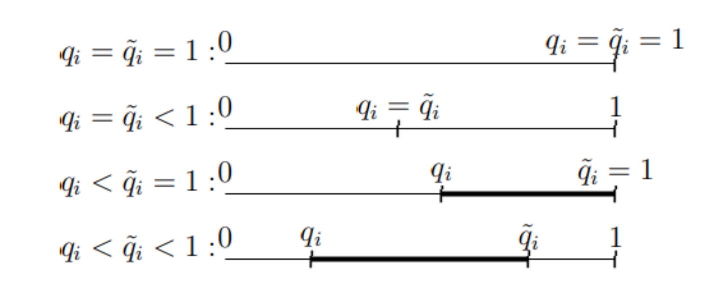

Figure: Visualizatin for continuum of fixed points (B-H '17)

Open questions and conjectures

Open questions (Bertacchi and Zucca '20):

- If there is any possibility for $|Q| < |T(\boldsymbol{f})| < \infty$?
- If there is any possibility for Q and $T(\boldsymbol{f})\backslash Q$ are both infinite?
- Given a typeset A, how to locate $\boldsymbol{q}(A)$ in Q or $T(\boldsymbol{F})$?

Conjectures (Bertacchi and Zucca '20, Braunsteins and Sophia '20):

- Q (same for $T(\boldsymbol{F})$) is either finite or uncountable.
- If $\boldsymbol{q} < \tilde{\boldsymbol{q}}$, there are continuums in Q and $T(\boldsymbol{F})$, with minimal $\boldsymbol{q}$ and maximal $\tilde{\boldsymbol{q}}$.

Open questions and conjectures

Bertacchi et al. (2022) resolved part of these questions.

1. There are examples for any number of extinction probability vectors in irreducible cases.
2. If $q^{(i)}(\{i\}) = q^{(i)}$ for any i, then $Q = T(\boldsymbol{F})$.
3. In irreducible case, there is no fixed point between $\tilde{\boldsymbol{q}}$ and $\mathbf{1}$;
4. In irreducible case, if $\sup_i \tilde{q}^{(i)} < 1$, then $\boldsymbol{q} = \tilde{\boldsymbol{q}}$ and $Q = T(\boldsymbol{F}) = \{\boldsymbol{q}, \mathbf{1}\}$.
5. Sufficient and necessary conditions for $q^{(i)}(A) < q^{(i)}(B)$ for two typesets A and B.

Outline

Introduction

Consider a GWP-∞ $\{\boldsymbol{Z}_n; n \geq 0\}$ in which the generating function $\boldsymbol{F}(\boldsymbol{s}) = (F^{(1)}(\boldsymbol{s}), F^{(2)}(\boldsymbol{s}), \cdots)$ has the form as

$$F^{(i)}(\boldsymbol{s}) = \sum_{j_1, j_2, \cdots \geq 0} P(j_1, j_2, \cdots) \prod_{k=1}^{\infty} s_{i+k-1}^{j_k},$$

where $\boldsymbol{s} = (s_1, s_2, \cdots)$ and $P(j_1, j_2, \cdots)$ represents the probability of a particle of type i gives j_k offspring of type $i + k - 1$ for $k \geq 1$ respectively. Denote the mean matrix of $\{\boldsymbol{Z}_n; n \geq 0\}$ by $\boldsymbol{M} = ((m_{ik}))$ where

$$m_{ik} = \frac{\partial F^{(i)}}{\partial s_k}(\mathbf{1}).$$

Braunsteins and Hautphenne (2017) showed the relations between Q and $T(\boldsymbol{F})$ in Lower Hessenberg GWP-∞, which assume type i particles can only produce type $j \leq i+1$ particles.

In this model, type i particles can only produce type $j \geq i$ particles. What bout the relations between Q and $T(\boldsymbol{F})$?

For $k \geq 1$, define

$$M_k = \frac{\partial F^{(1)}}{\partial s_k}(\mathbf{1}) = m_{1k} \quad \text{and} \quad M = \sum_{k \geq 1} M_k.$$

Assumptions:

A1: For any $k \geq i > 0$, there exists a positive integer n such that $\boldsymbol{M}_{ik}^n > 0$.

A2: $P(\mathbf{0}) > 0$ and $\mathbb{P}(|\boldsymbol{Z}_1| > 1) > 0$.

A3: $M_1 < 1$ and $M < \infty$.

Main results

Theorem (Tan and Zhang '23+)

If $M \leq 1$, then $Q = T(\boldsymbol{F}) = \{\mathbf{1}\}$. If $M > 1$ and $\sum_{i=1}^{\infty} M_i M^{1-i} > 1$, then $T(\boldsymbol{F})$ has at least countably many fixed points while $Q = \{q\mathbf{1}, \mathbf{1}\}$, where $q < 1$ is an extinction probability.

Explanation

For an infinite-dimensional generating function

$$F^{(1)}(\boldsymbol{s}) = \sum_{\boldsymbol{j} \in l_1(\mathbb{N})} P(\boldsymbol{j}) \boldsymbol{s}^{\boldsymbol{j}},$$

if there exists $\boldsymbol{x}$ (except $q\mathbf{1}$ and $\mathbf{1}$) such that for any initial component i,

$$F^{(1)}(x_i, x_{i+1}, \cdots) = x_i.$$

The answer is positive and there exist at least countably infinitely many $\boldsymbol{x}$ with $(1 - x_i)/(1 - x_{i+1}) \to c \ (> 1)$.

Outline

Let $\boldsymbol{G}(s) = \mathbf{1} - \boldsymbol{F}(\mathbf{1} - \boldsymbol{s})$ and assume $M > 1$. Briefly speaking, we intend to find a sequence of infinite vectors $\{\boldsymbol{y}_n; n \geq 1\}$ such that there exists $\boldsymbol{y} \in (0,1)^{\mathbb{N}^+}$ with

$$|G^{(i)}(\boldsymbol{y}_n) - G^{(i)}(\boldsymbol{y})|, \ |G^{(i)}(\boldsymbol{y}_n) - y_n^{(i)}| \text{ and } |y_n^{(i)} - y^{(i)}|$$

converge to 0 respectively for any $i \geq 1$. Then by triangle inequality, $|G^{(i)}(\boldsymbol{y}) - y^{(i)}| = 0$ for any i.

Ideas of proof

At first, we make a paraphrasing for $F^{(1)}(\boldsymbol{s})$.

Let $h_0 = P(\mathbf{0})$ and h_i $(i \geq 1)$ be the probability that the offspring of a type 1 particle has the maximal type of i, that is,

$$h_i = \sum_{\substack{j_1,\cdots,j_{i-1}\geq 0\\ j_i>0}} P(j_1,\cdots,j_i,0,0,\cdots).$$

Define the k-dimensional probability generating function

$$f_k(s_1,\cdots,s_k) = \sum_{\substack{j_1,\cdots,j_{k-1}\geq 0\\ j_k>0}} \frac{P(j_1,\cdots,j_k,0,\cdots)}{h_k} \prod_{i=1}^{k} s_i^{j_i}.$$

Then

$$F^{(1)}(\boldsymbol{s}) = h_0 + \sum_{k=1}^{\infty} h_k f_k(s_1,\cdots,s_k).$$

Ideas of proof

From this paraphrasing we can use the following lemma to calculate $\mathbf{1} - \boldsymbol{F}(\mathbf{1} - \boldsymbol{s})$.

Lemma (Joffe '67)

For any finite-type generating function $\boldsymbol{L}(\boldsymbol{s})$ and its corresponding mean matrix $\boldsymbol{M}_0$, it holds that

$$\mathbf{1} - \boldsymbol{L}(\boldsymbol{s}) = (\boldsymbol{M}_0 - \boldsymbol{E}(\boldsymbol{s}))(\mathbf{1} - \boldsymbol{s}),$$

where $\mathbf{0} \leq \boldsymbol{E}(\boldsymbol{s}) \leq \boldsymbol{M}_0$ elementwise, $\boldsymbol{E}(\boldsymbol{s})$ is non-increasing in $\boldsymbol{s}$ (with respect to the partial order induced by $\leq$) and tends to $\mathbf{0}$ as $\boldsymbol{s} \to \mathbf{1}$.

Ideas of proof

Then we can prove that there exists a positive number $\gamma < 1$ which is actually the solution to the equation $\sum_{i=1}^{\infty} M_i s^{i-1} = 1$, such that for any $\boldsymbol{t} \in \mathcal{H}$, we have $\boldsymbol{G}(\boldsymbol{t}) \in \mathcal{H}$, where

$$\mathcal{H} = \{\boldsymbol{x} : \forall\ i > 0, x^{(i)} \in (0,1), x^{(i)} > x^{(i+1)} \text{ and } \lim_{i \to \infty} \frac{x^{(i+1)}}{x^{(i)}} = \gamma\}.$$

Find an arbitrary vector $\boldsymbol{y}_0 \in \mathcal{H}$. We can replace the front components of $\boldsymbol{y}_0$ by iterating on $\boldsymbol{G}(\boldsymbol{s})$ and retain the tail components to obtain a sequence $\{\boldsymbol{y}_n; n \geq 0\}$. It is clear that $\boldsymbol{y}_n \in \mathcal{H}$ and $\boldsymbol{G}(\boldsymbol{y}_n) \in \mathcal{H}$ for any n. Next, we prove that $|\boldsymbol{G}(\boldsymbol{y}_n) - \boldsymbol{y}_n| \to 0$ as $n \to \infty$. Finally, we show that $\boldsymbol{y}_n$ has a nondegenerating limit (pointwisely).

Further researches on GWP-∞

The current researches for GWP-∞ can be classified in three aspects:

- Consider a generic GWP-∞ to get universal criterion for Q, $T(\boldsymbol{F})$ (Bertacchi et al. '22);
- Consider the GWP-∞ with special transition probability, such as lower Hessenberg GWP-∞ and linear fraction GWP-∞ (Braunsteins '19, Sagitov '13);
- Consider the case of τ-recurrent, expand classical theorems in GWP to GWP-∞ (Moy '67, Vatutin '22).

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Thank you!